JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 18, No. 6, November–December 1995

# Observability of Gravimeter-Aided Inertial Navigation Systems

Y. G. Prokhorov*
*State Academy of Management, Moscow 109444, Russia*

**In this work, the possibility of creating inertial navigation systems corrected in the presence of gravity anomalies is investigated. Observability of gravimeter-aided inertial navigation systems is an important practical question when Kalman filtering is applied for its correction. To solve this problem, error equations are obtained in which errors stipulated by geocentric and geodetic noncoincidence of triads during a vehicle motion above anomaly of the level surface are taken into account. The observability of inertial navigation systems with a gravimeter in the vertical channel is proved in this article. The conditions for observability of position, velocity, and drifts of a gyrostabilized platform on a moving vehicle are obtained.**

## Nomenclature

$a, b$ = lengthwise and cross gradients of vertical component of anomalous Earth's gravitational field
$r$ = distance from Earth center to moving vehicle
$\lambda$ = geocentric longitude
$\varphi$ = geocentric latitude
$\omega$ = angular velocity of platform of which GINS is mounted
$\omega_0$ = Schuler frequency, $\omega_0^2 = \mu/r^3$

## Introduction

THE conditions for observability inertial navigation systems (INS) are obtained when correction is accomplished by means of comparison of the information about Earth's gravitational field measured by a vertical gravimeter with map values of gravity anomalies stored in an onboard memory of a moving vehicle. A gravimeter-aided inertial navigation system (GINS) is observable during vehicle motion with constant velocity along the equator at invariable (constant) distance from the motionless theoretical Earth in the form of a spheroid if and only if[1] rank $S = 7$, where $S$ is the observability matrix. In this assumption it can be shown that

$$\det(S) = 4ab^4r^3\omega_0^2\omega^2\left(\omega_0^2 - \omega^2\right)\left(\omega_0^4 - 4\omega^4\right) \neq 0 \tag{1}$$

From the inequality (1) it follows that GINS is unobservable when a vehicle is motionless ($\omega = 0$) or when it moves at velocity $\omega = \omega_0/\sqrt{2}$ or when $\omega = \omega_0$.

## Nominal and Error Equations

Make the following simplified assumptions:

1) Earth is considered to be motionless. The origin of the right-hand coordinate system $A\xi\eta\zeta$ is to be fixed in the Earth center $A$. Define the moving vehicle position point $O$ by the radius vector as

$$\bar{r} = \xi\bar{\xi} + \eta\bar{\eta} + \zeta\bar{\zeta} \tag{2}$$

where $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ are the corresponding axis unit vectors. Then the vehicle velocity is

$$\bar{V} = \frac{d\bar{r}}{dt} \tag{3}$$

Introduce geocentric coordinates so that

$$\xi = r\cos(\varphi)\cos(\lambda), \qquad \eta = r\cos(\varphi)\sin(\lambda), \qquad \zeta = r\sin(\varphi) \tag{4}$$

Introduce the $O_{xyz}$ geocentric coordinate grid $O_x, O_y, O_z$ axes are aligned to east, north, vertical to the spherical Earth's surface) and the $A_{xyz}$ triad at the origin in the center of Earth and with axes oriented analogously to the $O_{xyz}$ triad. If the projection of the radius vector $\bar{r}$ on the $A_{xyz}$ axes is $\bar{r} = (r_x, r_y, r_z)$, then $r_x = r_y = 0$ and $r_z = r$.

2) The normal Earth's gravitational field is considered to be spherical with intensity in the point defined by radius vector $\bar{r}$,

$$\bar{g}_n = -(\mu\bar{r}/r^3) \tag{5}$$

where $\mu = MG$, $M$ is the Earth's mass ($M = 5.98 \times 10$ kg), and $G$ is the gravitational constant ($G = 6.67 \times 10^{-11}$ m$^3$/kg $\cdot$ s$^2$).

3) Define the vertical component of gravitational anomaly as follows:

$$g_a = \bar{g}_a\bar{z} \tag{6}$$

Here $\bar{z}$ is the basis vector ($\bar{z} = \bar{r}/r$).

4) State-space accelerometer output projected in the inertial coordinate system $A\xi\eta\zeta$ is

$$\bar{n} = n_\xi\bar{\xi} + n_\eta\bar{\eta} + n_\zeta\bar{\zeta} = \frac{d\bar{v}}{dt} + \frac{\mu\bar{r}}{r^3} + \bar{g}_a \tag{7}$$

5) With the above assumptions, the nominal equations[1–3] become

$$\frac{d}{dt}(\bar{r}\times\bar{v}) = \bar{r}\times\bar{n}, \qquad \bar{v} = \frac{d\bar{r}}{dt}, \qquad r = r(t)$$
$$g_a = -\frac{\bar{r}}{r}\left(\bar{n} - \frac{d\bar{v}}{dt} - \frac{\mu\bar{r}}{r^3}\right) \tag{8}$$

The standard procedure for obtaining error equations is based on perturbation equations (8), ignoring nonlinear terms. The following vector expressions are its result:

$$\bar{r}\times\left[\frac{d\,\delta\bar{v}}{dt} + \left(\frac{\mu}{r^2} - g_a\right)\frac{\delta\bar{r}}{r}\right] = \bar{r}\times(\Delta\bar{n} - \bar{\theta}\times\bar{n}) \tag{9a}$$

$$\frac{d\,\delta\bar{r}}{dt} = \delta\bar{v}, \qquad \frac{d\bar{\theta}}{dt} = \Delta\bar{\omega}, \qquad \Delta r_z = \Delta r_z(t) \tag{9b}$$

$$\delta g_a = \left(\frac{\delta\bar{r}}{r} - \frac{\Delta r_z\bar{r}}{r^2}\right)\left[\bar{n} - \frac{d\bar{v}}{dt} - \left(\frac{\mu}{r^2} - g_a\right)\frac{\bar{r}}{r}\right]$$
$$-\frac{\bar{r}}{r}\left(\Delta\bar{n} + \bar{\theta}\times\bar{n} - \frac{d\,\delta\bar{v}}{dt} - \frac{\mu\,\delta\bar{r}}{r^3} + \frac{3\mu\bar{r}\,\Delta r}{r^4}\right) \tag{9c}$$

Where the symbol $\delta$ means variation of the following value, $\Delta r_z$ is the altitude sensor error, $\bar{\theta} = (\theta_x, \theta_y, \theta_z)$ is the vector error of

Received April 6, 1992; revision received April 14, 1994; accepted for publication Jan. 16, 1995. 

*Associate Professor, Department of Applied Mathematics.

the frame coordinate system orientation, $\Delta\bar{\omega}$ is the gyroscope drift vector, and $\Delta\bar{n}$ is the accelerometer instrumental error vector.

## INS Correction According to Gravity Anomalies

Under the perturbing work by a vertical gravimeter the value is measured:

$$g_a'(\bar{r}) = \bar{g}_a'\bar{z} = [\bar{g}_a(\bar{r}) + \delta\bar{g}_a]\bar{z} = \bar{g}_a(\bar{r})\bar{z} + \delta\bar{g}_a\bar{z} \qquad (10)$$

Here $\bar{z}$ is the axes $O_z$ unit vector; the member $\delta\bar{g}_a = \delta g_a\bar{z}$ is defined by expression (9c).

The quantity $g_a'(\bar{r})$ is compared with the value $g_a(\bar{r} + \delta\bar{r})$ of an onboard gravity anomaly map at position $\bar{r} + \delta\bar{r}$, that is, with an INS position error $\delta\bar{r}$.

The observing value under the INS correction is the following:

$$Z(t) = [g_a'(\bar{r}) - \bar{g}_a(\bar{r} + \delta\bar{r})]\bar{z} = \bar{g}_a'(\bar{r})\bar{z} - \bar{g}_a(\bar{r} + \delta\bar{r})\bar{z} \qquad (11)$$

Here $g_a(\bar{r} + \delta\bar{r}) = g_a(\bar{r} + \delta\bar{r})\bar{z}$.

Consider the smoothed-out function $g_a(\bar{r})$, which is differentiated on $\varphi$, $\lambda$, $r$ and limited by line terms. Restricting its equation to linear terms only gives the following:

$$g_a(\bar{r} + \delta\bar{r}) = g_a(\bar{r}) + \frac{1}{r}\frac{\partial g_a}{\partial\varphi}\delta r_y + \frac{1}{r\cos\varphi}\frac{\partial g_a}{\partial\lambda}\delta r_x + \frac{\partial g_a}{\partial r}\Delta r_x \qquad (12)$$

In conjunction with the uniform vehicle motion along Earth's equator at a constant distance from Earth ($\omega_x = \omega_z = 0$; $\omega_y =$ const, $r =$ const) after introducing the terms

$$a = \frac{\partial g_a}{\partial\lambda r\cos\varphi}, \qquad b = \frac{1}{r}\frac{g_a}{\partial\varphi}, \qquad c = \frac{\partial g_a}{\partial r}$$

the observable value expression becomes

$$Z(t) = 2\omega_y\,\delta V_x - a\,\delta r_x - b\,\delta r_y - c\,\Delta r_z \qquad (13)$$

It can be easily shown that $\delta r_x = \delta\varphi/r$, $\delta r_y = \delta\lambda/r$, $\Delta r_z = 0$, and horizontal gradients are $a = \partial g_a/\partial r_x$, $b = \partial g_a/\partial r_y$.

## Model of Level Surface of Earth's Gravitational Field, Taking into Account Anomalous Mass

Examine a vehicle motion above the level surface of Earth's gravitational field, taking into account the anomalous mass. Earth's surface is a complex surface—geoid, which is close to the physical Earth's surface. Place a gravity anomaly in the form of point mass $m_a$ at the depth $H$ of the theoretical Earth.

At any point $O$ outside Earth the gravitational potential can be represented as a sum of normal and anomalous potentials $V_n + V_a = \mu/r + \mu_a/\rho$, where $\mu_a = m_a G$, and $\rho$ is a distance from the anomaly mass location to any point $O$ where a vehicle is located.

The point mass method allows us to define the level surface equation to calculate the vertical anomalous Earth's gravitational field $g_a$ and also to derive lengthwise and cross gravity gradients $g_a$, i.e., $a = \partial g_a/\partial r_x$, $b = \partial g_a/\partial r_y$.

With restriction to small variations of geocentric coordinates ($\varphi \le 1$ deg, $\lambda \le 1$ deg) and neglecting higher than the second order of these quantities ($\sin\varphi \approx \varphi$, $\cos\varphi \approx 1 - \frac{1}{2}\varphi^2$, $\sin\lambda \approx \lambda$, $\cos\lambda \approx 1 - \frac{1}{2}\lambda^2$) it can be shown that

$$r = (R + d) \times \left\{1 + \frac{m_a}{M}\frac{R + h}{H + r}\left[1 + \frac{1}{2}\frac{(R + h)(R - H)}{(H + h)^2}(\varphi^2 + \lambda^2)\right]\right\} \qquad (14)$$

$$g_a(\bar{r}) = -\frac{\mu_a}{(H + h)^2} + \frac{1}{2}K(R + h)(\varphi^2 + \lambda^2) \qquad (15)$$

$$a = \frac{\partial g_a}{\partial r_x} = K\lambda, \qquad b = \frac{\partial g_a}{\partial r_y} = K\cdot\varphi \qquad (16)$$

where $R$ is Earth's radius ($R = 6.37 \times 10^6$ m) and $h$ is the height above the theoretical Earth defining the family of level surfaces. Geoid corresponds to height $h = 0$; $K = 3\mu_a(R - H)(R + H)/(H + h)^4$.

Assume that point mass $m_a = 7.5 \times 10^7$ kg is placed at the depth $H = 10^5$ m of the theoretical Earth. Then the maximum value of the vertical component of the gravity anomaly on the geoid surface ($h = 0$) is $g_a \approx 500$ milliGal, and $g_a \approx 410$ milliGal at the height $h = 10^4$ m. The horizontal gradient $g_a(\bar{r})$ averages 2–4 milliGal/km in the vicinity of the point $\lambda = 0$, $\varphi = 0$ ($\le 100$ km). All this is consistent with gravitational expected data of some real anomalies.[4]

## Correction Model in Geodetic Coordinate Grid

Let the vehicle be located at any point $O$ in the $A\xi\eta\zeta$ coordinate system. Draw the normal to the real level surface through this point. It will cross the geoid at point $B$- point $O$ location may be determined by geodetic latitude $\varphi'$, geodetic longitude $\lambda'$, and normal segment $h$ between point $O$ and $B$ (Figs. 1 and 2).

Set a correspondence of geodetic coordinates $\varphi' = 0$, $\lambda' = 0$, to the above anomaly. Introduce geodetic coordinates for any point $O$. The angle $\varphi'$ is formed by the normal $A'O$ to the level surface and

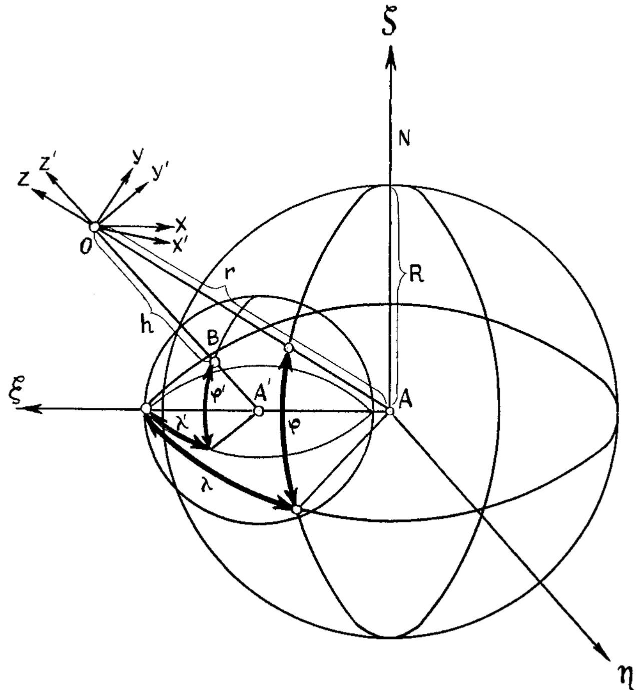

**Fig. 1 Geodetic and geocentric coordinate grids.**

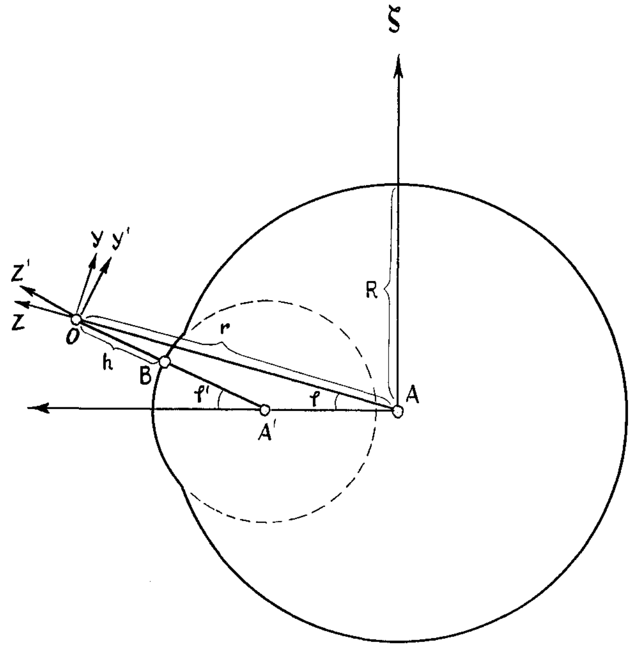

**Fig. 2 Geodetic and geocentric coordinate latitudes.**

equator plane. The angle $\lambda'$ is formed by a normal plane passed through point $O$ to be orthogonal with the equator plane and the zero meridian plane (Figs. 1 and 2).

The reciprocal axis orientation of geocentric ($O_{XYZ}$) and geodetic ($O_{X'Y'Z'}$) triads is defined by the relationship

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = D \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (17)$$

where the direction cosine matrix is

$$D = \begin{bmatrix} \cos(\lambda' - \lambda) & 0 & -\sin(\lambda' - \lambda) \\ -\sin(\lambda' - \lambda)\cos(\varphi' - \varphi) & \cos(\varphi' - \varphi) & \sin(\varphi' - \varphi)\cos(\lambda' - \lambda) \\ \sin(\lambda' - \lambda)\cos(\varphi' - \varphi) & \sin(\varphi' - \varphi) & \cos(\lambda' - \lambda)\cos(\varphi' - \varphi) \end{bmatrix}$$

When a vehicle moves above the anomaly, Earth's gravitational field will be measured in the geocentric $O_{XYZ}$ coordinate system, but the results will relate to the geodetic $O_{X'Y'Z'}$ triad. Project vector error (9a, b) alignments on geodetic $O_{X'Y'Z'}$ axes according to the direction cosine matrix (17). View the particular case of vehicle uniform motion along geodetic parallel $\varphi' = \Delta\varphi' = \text{const}$ to be close to the geodetic equator (i.e., $\Delta\varphi' \approx 0$ deg) at a constant distance from the geoid. Then the matrix form of the error equations under small variations of angle $\lambda'$ (i.e., $\sin\lambda' \approx \lambda'$, $\cos\lambda' \approx 1 - \lambda'^2/2$) can be written as

$$\dot{X} = AX + U(t) \quad (18)$$

where

$$A = \begin{bmatrix} 0 & 0 & -\left[\omega_0^2(1 - 3\,\mathrm{d}^2\lambda'^2) - \omega_{y'}^2\right] & 0 & 0 & -r\omega_0^2\left(1 - \frac{\mathrm{d}^2\lambda'^2}{2}\right) + R_1\omega_{y'}^2 & -r\omega_0^2\,\mathrm{d}\lambda \\ 0 & 0 & 0 & \omega_0^2 & r\omega_0^2\left(1 - \frac{\mathrm{d}^2\lambda'^2}{2}\right) - R_1\omega_{y'}^2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega y' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +\omega y' & 0 & 0 \end{bmatrix}$$

$$X = \|\delta V_{x'}; \delta V_{y'}; \delta r_{x'}; \delta r_{y'}; \theta_{x'}; \theta_{y'}; \theta_{z'}\|^T$$

Here the symbol $T$ means the transpose of a matrix,

$$U = \|\Delta n_{x'}; \Delta n_{y'}; 0; 0; \Delta\omega_{x'}; \Delta\omega_{y'}; \Delta\omega_{z'}\|^T$$

Here $r = R + h$:

$$d = \frac{R + h - R_1 + (m_a/M)[(R+h)^2/(H+h)]}{R + h + (m_a/M)[(R+h)^2/(H+h)]}$$

On the geoid surface ($h = 0$) the coefficient $d \approx \frac{1}{4}$; $R_1$ is the radius of the level surface curvature in a small vicinity ($\leq 100$ km) of point ($\varphi' = 0$; $\lambda' = 0$) where the gravity anomaly mass $m_a$ is concentrated:

$$R_1 = \frac{R + h}{1 + (m_a/M)[(R+h)^3/(H+h)^3]}$$

By projection of the difference vector $\bar{g}'_a - \bar{g}_a(\bar{r} + \delta\bar{r})$ on the $O_{Z'}$ axis of the geodetic triad $O_{x'y'z'}$, the observable value under correction can be obtained:

$$Z(t) = [\bar{g}'_a - \bar{g}_a(\bar{r} + \delta\bar{r})]\bar{z}' = CX + V(t) \quad (19)$$

Here the matrix

$$C = \left\|2\omega y'; 0; a(1 - \mathrm{d}^2\lambda'^2) + 3\,\mathrm{d}\lambda'; b\left(1 - \tfrac{1}{2}\mathrm{d}^2\lambda'^2\right); r\omega^2\,\mathrm{d}\lambda'; 0; 0\right\|$$

and the measurement noise is $V(t) = \|\Delta n_z\|$. Lengthwise and cross gradients $a$, $b$ are determined by relation (16).

The characteristic polynomial of matrix $A$,

$$\det(\nu E - A)' = \nu\left(\nu^2 + \omega_{y'}^2\right)\left(\nu^2 + \omega_0^2\right) \times \left[\nu^2 + \omega_0^2(1 - 3\,\mathrm{d}^2\lambda'^2) - \omega_{y'}^2\right] \quad (20)$$

has prime roots $\nu_1 = 0$, $\nu_{2,3} = \mp i\omega y'$, $\nu_{4,5} = \mp i\omega_0$, $\nu_{6,7} = \mp i[\omega_0^2(1 - 3\,\mathrm{d}^2\lambda'^2) - \omega_y^2]^{1/2}$.

Equation (20) is the minimum polynomial of the matrix $A$. In this connection the observability matrix of system (18) and (19) becomes

$$S = \|C^T; A^TC^T; (A^T)^2C^T; (A^T)^3C^T; (A^T)^4C^T; (A^T)^5C^T; (A^T)^6C^T\| \quad (21)$$

The rank $(S) = 7$ is a necessary and sufficient condition of GINS observability. It is equivalent to the inequality

$$\det(S) = 4[K(1-d)\lambda'(1 - \mathrm{d}^2\lambda'^2) + 3\,\mathrm{d}\lambda'] \times \left[K(1-d)\,\Delta\varphi'\left(1 - \tfrac{1}{2}\mathrm{d}^2\lambda'^2\right)\right]^4\left[r\omega_0^2(1 - 3\,\mathrm{d}^2\lambda'^2) - \omega_{y'}^2\right] \times \left\{\left(1 - \tfrac{1}{2}\mathrm{d}^2\lambda'^2\right) - \omega_{y'}^2\right\}r^2 \neq 0 \quad (22)$$

## Discussion of Results

In the GINS correction model of the second approximation, viewed in this article, critical points ($\omega = 0$, $\omega = \omega_0/\sqrt{2}$, $\omega = \omega_0$) do not lead to INS unobservability as shown by formula (22). Moreover, since the geodetic longitude changes when a vehicle moves, we can indicate zeros of the above-cited polynomial. But these zeros do not mean unobservability conditions; they only show the change of sign of the ratio (22).

When vehicle velocities are close to zero ($\omega y' \leq 0.25 \times 10^{-5}\ \mathrm{s}^{-1}$), the INS becomes weakly observable only in the immediate vicinity of the point ($\varphi' = 0, \lambda' = 0$). The system is unobservable for a motionless vehicle ($\omega y' = 0$) only at the point ($\varphi' = 0, \lambda' = 0$). These circumstances exclusively show that the given point is above the anomaly center of symmetry when gravity anomaly mass $m_a$ is placed at the point ($\varphi' = 0, \lambda' = 0$) at the depth $H$ of the theoretical Earth. As a consequence of this symmetry, the lengthwise and cross gradients of the anomalous Earth's gravitational field vertical components are equal to zero at this point, i.e., $a = b = 0$.

## Conclusions

Gradients of the vertical component of the gravity anomaly are not equal to zero at any vehicle position is a necessary condition of GINS observability. Then, the system is unobservable if either gradient (lengthwise or cross) is equal to zero. This is a sufficient condition of GINS unobservability.

In conclusion, it is necessary to note that the models and the formula dependences in this article solve an important problem of gravity anomaly mapping from a moving vehicle. They are of importance for providing not only GINS operation but also systems that can be used to study a geological Earth structure and for minerals prospecting.

## References

[1]Bar-Itzhack, I. Y., and Berman, N., "Control Theoretic Approach to Inertial Navigation Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 11, No. 3, 1988, pp. 237–245.

[2]Bar-Itzhack, I. Y., and Berman, N., "Identity between INS Position and Velocity Error Models," *Journal of Guidance, Control, and Dynamics*, Vol. 11, No. 6, 1988, pp. 590–592.

[3]Pitman, G. R., *Inertial Guidance*, New York, 1961.

[4]Heiskanen, W. A., and Meinesz, F. A., *The Earth and Its Gravity Field*, New York, 1958.